BIT
2000, Vol., No., pp. 1–10

0006-3835/98/3804-0101 $12.00

COMPUTATION OF BESSEL AND AIRY FUNCTIONS AND OF RELATED GAUSSIAN QUADRATURE FORMULAE *

WALTER GAUTSCHI[1]

[1]Department of Computer Sciences, Purdue University, West Lafayette, IN 47907, USA. email: wxg@cs.purdue.edu

Abstract.

Procedures are described for the high-precision calculation of the modified Bessel function $K_\nu(x)$, $0 < \nu < 1$, and the Airy function $\mathrm{Ai}(x)$, for positive arguments x, as prerequisites for generating Gaussian quadrature rules having these functions as weight function.

AMS subject classification: 33C10, 33C45, 65D20, 65D32.

Key words: Modified Bessel function, Airy function, Gaussian quadrature.

1 Introduction.

Integrals involving modified Bessel functions K_ν for $\nu = \frac{1}{3}$ and the Airy function Ai occur naturally in some physics applications (see, e.g., Gordon [5], [6]); the weight function K_0 has also found use in the asymptotic estimation of oscillatory integral transforms (see Wong [12], Gautschi [4, Example 6.1, p. 94]). Efficient evaluation of such integrals calls for Gaussian quadrature rules having these functions as weight function. Such rules (involving up to 10 points) were already given in [6] for the modified Bessel function $K_{\frac{1}{3}}$ and in [5], [9] for the Airy function. It appears, however, that the latter are in error because of an incorrect calculation of the relevant moments. Here we develop Gaussian quadrature formulae for both weight functions with up to $n = 40$ points. The main task is to find the first n recursion coefficients in the three-term recurrence relation for the relevant orthogonal polynomials. These can be obtained to arbitrary precision from the known moments by symbolic computation, or else, to standard machine precisions, by general procedures developed earlier in [3]. To apply these procedures, it is important to have routines that calculate modified Bessel functions and the Airy function to high accuracy. Such routines are described in §2 for Bessel functions, and in §3 for the Airy function. Section 4 discusses the computation of the respective Gaussian quadrature rules. Two appendices contain the first 40 of the requisite recursion coefficients to 28 decimals.

*Received … . Communicated by … .

2 Modified Bessel functions.

Our interest is in generating values to high accuracy of the modified Bessel function $K_\nu(x)$, where $0 < \nu < 1$ and $x > 0$, as a preparation for generating the Gaussian quadrature rules in §§4.1 and 4.2. A case of particular interest is $\nu = \frac{1}{3}$, but the procedure we develop is applicable also for other values of ν except those close to 0 and 1. It is not our intention, here, to develop a general-purpose production code. There are a number of such codes available in the literature (see, e.g., [8, §§4.1, 5.1]), that — like ours — are suitable also for high-accuracy work. What we find worth observing is the apparently novel use of integral representations and related generalized Gauss-Laguerre quadratures to compute modified Bessel functions for moderately large, and large, real arguments. The approach, in fact, is potentially useful also for complex arguments.

When x is relatively small, say $0 < x \le 2$, we use, as others have done before, the representation (cf. [1, Eq. 9.6.2])

$$K_\nu(x) = \tfrac{1}{2}\pi\, \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \pi\nu}\,, \tag{2.1}$$

and evaluate $I_{\pm\nu}(x)$ by Taylor expansion ([1, Eq. 9.6.10])

$$I_{\pm\nu}(x) = (\tfrac{1}{2}x)^{\pm\nu} \sum_{k=0}^{\infty} \frac{(\frac{1}{4}x^2)^k}{k!\Gamma(k+1\pm\nu)}\,. \tag{2.2}$$

If ν is close to 0 or 1, considerable cancellation occurs in the numerator of (2.1). This could be dealt with, if deemed necessary, by special additional procedures (cf., e.g., [10, Sec. II]). Here we simply assume $.05 \le \nu \le .95$, which limits the loss of accuracy to at most two (or three, if x is near 2) decimal digits.

For $x > 2$, we use the integral representation ([1, Eq.9.6.23])

$$K_\nu(x) = \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+\frac{1}{2})}\, \frac{e^{-x}}{\sqrt{x}} \int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t}\, dt, \tag{2.3}$$

where the integral is conveniently evaluated by generalized Gauss-Laguerre quadrature with parameter $\alpha = \nu - \frac{1}{2}$,

$$\int_0^\infty \left(2 + \frac{t}{x}\right)^{\nu-\frac{1}{2}} \cdot t^{\nu-\frac{1}{2}} e^{-t}\, dt \simeq \sum_{k=1}^{n} w_k^L \left(2 + \frac{t_k^L}{x}\right)^{\nu-\frac{1}{2}}, \quad x > 2. \tag{2.4}$$

Here, t_k^L, w_k^L are the nodes and weights of the generalized Gauss-Laguerre quadrature rule. (The dependence on n is suppressed in the notation.) These, for $\nu = \frac{1}{3}$, were generated by double-precision resp. quadruple-precision analogues of the procedures `recur` and `gauss` of [3]. While it would be unreasonable to expect convergence to full machine precision as $n \to \infty$, it was found that in double and quadruple precision, "numerical convergence" occurs to an accuracy of $10\times\varepsilon_{\text{dble}}$ resp. $1000\times\varepsilon_{\text{quad}}$, where $\varepsilon_{\text{dble}} \simeq .111 \times 10^{-15}$, $\varepsilon_{\text{quad}} \simeq .963 \times 10^{-34}$ are respectively the IEEE double- and quadruple-precision machine precisions.

In other words, the approximants stabilized to these accuracies at certain values of n, which are shown in Table 2.1.

Table 2.1: Number of Gauss points in (2.4), with $\nu = \frac{1}{3}$, required for double- and quadruple-precision accuracy.

ν	0	1/6	1/3	1/2	2/3	5/6	1
n double	23	23	22	1	21	22	22
n quadruple	87	85	83	1	82	82	82

(The data is for $x = 2$; as x increases, n decreases.)

In many applications (including the one in §4.1; cf. (4.7)), it is better to compute $e^x K_\nu(x)$. This is also the function tabulated (for $\nu = \frac{1}{3}$) in [11, Table III].

We remark that our evaluation procedure is also applicable for x in the complex plane cut along the negative real axis, but we will not pursue this here any further.

3 The Airy function.

The Airy function is related to the modified Bessel function $K_{\frac{1}{3}}$ as follows (cf. [1, Eq. 10.4.14]):

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}\, K_{\frac{1}{3}}(\zeta), \quad \zeta = \tfrac{2}{3}x^{\frac{3}{2}}. \tag{3.1}$$

Using (2.1) with $\nu = \frac{1}{3}$, one gets

$$\mathrm{Ai}(x) = \frac{1}{3}\sqrt{x}\,[I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)]. \tag{3.2}$$

Here, for $0 < \zeta \le 2$, i.e., $0 < x \le 3^{2/3} = 2.08008\ldots$, both Bessel functions can be evaluated by Taylor expansion as in (2.2) with $\nu = \frac{1}{3}$.

For $\zeta > 2$, we use the integral representation (2.3) for $K_{\frac{1}{3}}$ in conjunction with (3.1) to obtain

$$\mathrm{Ai}(x) = \frac{1}{\sqrt{\pi}}\,\frac{\zeta^{-\frac{1}{6}}e^{-\zeta}}{(48)^{\frac{1}{6}}\Gamma(\frac{5}{6})}\int_0^\infty \left(2 + \frac{t}{\zeta}\right)^{-\frac{1}{6}} \cdot t^{-\frac{1}{6}}e^{-t}\,dt, \tag{3.3}$$

with ζ as defined in (3.1). Now generalized Gauss-Laguerre quadrature is appropriate with Laguerre parameter $\alpha = -\frac{1}{6}$. According to Table 2.1 ($\nu = \frac{1}{3}$), a 22-point formula yields double-precision accuracy and a 83-point formula quadruple-precision accuracy.

This procedure can be used also for complex x, at least in the sector $|\arg x| < \frac{2}{3}\pi$.

4 Gauss quadratures.

4.1 Gauss quadrature with Bessel weight function.

We define the weight function

$$w(x) = \frac{2}{\pi}\cos(\tfrac{1}{2}\nu\pi)K_\nu(x), \quad 0 < x < \infty. \tag{4.1}$$

Its moments can be calculated explicitly by (cf. [7, 6.561.16])

$$\mu_k = \int_0^\infty x^k w(x)dx = \frac{1}{\pi}\cos(\tfrac{1}{2}\nu\pi)\cdot 2^k\Gamma(\tfrac{1}{2}(k+1+\nu))\Gamma(\tfrac{1}{2}(k+1-\nu)). \tag{4.2}$$

In particular, for $k = 0$,

$$\mu_0 = \frac{1}{\pi}\cos(\tfrac{1}{2}\nu\pi)\cdot\Gamma(\tfrac{1}{2}(1+\nu))\Gamma(\tfrac{1}{2}(1-\nu)) = 1 \tag{4.3}$$

by virtue of the reflection formula ([1, 6.1.17]) for $\Gamma(z)\Gamma(1-z)$, $z = \frac{1}{2}(1+\nu)$. Thus, (4.1) is a normalized weight function.

In principle, the first $2n$ moments μ_k of w can be used to generate the n-point Gauss formula for w. It is well known, however, that this becomes quickly unstable as n increases. A way around this problem is to employ a symbolic computation package combined with extended-precision arithmetic. A Maple script, named `cheb.mws`[1], has been developed for this purpose and has been used to produce the required recursion coefficients α_k, β_k (cf. (4.4) below) for $0 \le k \le 39$ to as many as 100 decimal digits[2]. It can be accessed at `http://www.cs.purdue.edu/archives/2001/wxg/codes/cheb.mws`. (A text file `cheb.txt` can be found at the same URL.)

Here, we describe a stable numerical procedure — a four-pronged discretization procedure (cf. [4, §6]) — that discretizes the inner product for the weight function w and generates the corresponding discrete orthogonal polynomials. If the discretization is chosen judiciously, the discrete orthogonal polynomials converge to the desired ones as the discretization is made increasingly finer. The first n recursion coefficients α_k, β_k in the three-term recurrence relation

$$\begin{gathered}\pi_{k+1}(x) = (x-\alpha_k)\pi_k(x) - \beta_k\pi_{k-1}, \quad k = 0,1,\ldots,n-1,\\ \pi_0(x) = 1, \quad \pi_{-1}(x) = 0,\end{gathered} \tag{4.4}$$

for the monic orthogonal polynomials (where $\beta_0 = \int_0^\infty w(x)dx$) are computed by the Stieltjes procedure ([4, §6.3]).

The discretization we choose makes use of a composition of the positive real axis into three subintervals, $\mathbb{R}_+ = (0, x_0] \cup [x_0, x_1] \cup [x_1, \infty)$, with x_0, x_1 still to be selected such that $0 < x_0 \le 1$, $1 < x_1 < \infty$. In the first subinterval,

[1] This is written for Maple.Release 5.

[2] The author is indebted to Mr. Oscar Chinellato at the Institute for Scientific Computing of the ETH Zurich, Switzerland, for translating a slightly edited version of our ORTHPOL routine `cheb` (cf. [3]) into a Maple script.

the behavior of $K_\nu(x)$ for small x must be properly accounted for. One has ([1, Eqs. 9.6.2 and 9.6.10])

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\left\{\frac{(\frac{1}{2}x)^{-\nu}}{\Gamma(1-\nu)}S_{-\nu}(x) - \frac{(\frac{1}{2}x)^{\nu}}{\Gamma(1+\nu)}S_{\nu}(x)\right\}, \tag{4.5}$$

where

$$S_{\pm\nu}(x) = \sum_{k=0}^{\infty}\frac{(\frac{1}{4}x^2)^k\Gamma(1\pm\nu)}{k!\Gamma(k+1\pm\nu)}.$$

The two distinct behaviors $x^{-\nu}$ and x^{ν} as $x \to 0$ need to be treated separately for purposes of integration. Indeed, the following composition for integrals over $(0, x_0]$ is suggested:

$$\begin{aligned}\int_0^{x_0} p(x)K_\nu(x)dx &= \frac{2^{\nu-1}\pi}{\sin\nu\pi\cdot\Gamma(1-\nu)}\int_0^{x_0} p(x)S_{-\nu}(x)\cdot x^{-\nu}dx \\ &- \frac{\pi}{2^{\nu+1}\sin\nu\pi\cdot\Gamma(1+\nu)}\int_0^{x_0} p(x)S_\nu(x)\cdot x^{\nu}dx.\end{aligned} \tag{4.6}$$

The first integral on the right is approximated by an N-point Gauss-Jacobi quadrature rule relative to the interval $(0, x_0]$ with Jacobi parameters $\alpha = 0$, $\beta = -\nu$, the second by a similar N-point Gauss-Jacobi rule with parameters $\alpha = 0$, $\beta = \nu$. In the second interval, we apply the ordinary N-point Gauss-Legendre rule transformed to the interval $[x_0, x_1]$. For the last interval, we write

$$\int_{x_1}^{\infty} p(x)K_\nu(x)dx = e^{-x_1}\int_0^{\infty} p(x_1+t)[e^{x_1+t}K_\nu(x_1+t)]\cdot e^{-t}dt \tag{4.7}$$

and approximate the integral on the right by an N-point Gauss-Laguerre quadrature rule, the function $e^x K_\nu(x)$, $x \geq x_1$, being computed from the integral representation (2.3) as discussed in the text following (2.3).

We call this a "four-pronged" discretization procedure since four different quadrature rules are employed to discretize the integral $\int_0^\infty p(x)K_\nu(x)dx$ (and with it the inner product relative to the weight function (4.1)): two Gauss-Jacobi rules for approximating the two integrals on the right of (4.6), the Gauss rule over $[x_0, x_1]$, and the Gauss-Laguerre rule to deal with the integral on the right of (4.7).

The parameters x_0, x_1 are chosen in an attempt to reduce the number N of quadrature terms required to achieve a given accuracy. Some limited experimentation suggested the choice $x_0 = 1$ and $x_1 = 10$. When $\nu = \frac{1}{3}$ and $n = 10$, then $N = 41$ will yield a relative accuracy of 1000 times the double machine precision (about 12 decimal-digit accuracy), and $N = 71$ a relative accuracy of 10^5 times the quadruple machine precision (about 29 decimal digits). For $n = 40$ the respective numbers are both $N = 81$.

The values of the recursion coefficients α_k, β_k (for $\nu = \frac{1}{3}$) to 28 decimal digits are given in Appendix A1 for $k = 0, 1, \ldots, 39$. These allow us to generate the respective Gauss and Gauss-Radau quadrature rules for up to 40 points by well-known eigenvalue/eigenvector techniques ([4, §4]). Double- and quadruple-precision fortran programs producing these coefficients are accessible at `http://www.cs.purdue.edu/archives/2001/wxg/codes/` in the files `dOPbess.f` and `qOPbess.f`. The coefficients themselves to 28 decimals can be found in the file `coeffbess` at the same URL. The file `ORTHPOLq` contains the quadruple-precision routines of the package ORTHPOL in [3].

4.2 Gauss quadrature with Airy weight function.

We define a weight function proportional to the one in Eq. (1.4) of [9],

$$w(x) = \frac{2^{\frac{2}{3}}\pi}{3^{\frac{5}{6}}\Gamma(\frac{2}{3})} x^{-\frac{2}{3}} e^{-x} \mathrm{Ai}((\tfrac{3}{2}x)^{\frac{2}{3}}), \quad 0 < x < \infty. \tag{4.8}$$

By (3.1), one has

$$w(x) = \frac{2^{\frac{1}{3}}}{3\Gamma(\frac{2}{3})}\, x^{-\frac{1}{3}} e^{-x} K_{\frac{1}{3}}(x). \tag{4.9}$$

As in [9], we use [7, 6.621.3] to obtain for the moments (correctly)

$$\mu_k = \int_0^\infty x^k w(x)dx = \frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma(\frac{2}{3})}\, \frac{k!\Gamma(k+\frac{1}{3})}{(6k+1)2^k\Gamma(k+\frac{1}{6})}. \tag{4.10}$$

In particular,

$$\mu_0 = \frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma(\frac{2}{3})}\, \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})} = 1 \tag{4.11}$$

by virtue of the duplication formula ([1, 6.1.19]) for $\Gamma(2z)$, $z = \frac{1}{6}$. Thus, the weight function (4.8) is normalized.

The three-term recurrence relation (4.4) for this weight function can again be obtained by symbolic computation using the Maple script `cheb.mws` (cf. §4.1). Numerically, on the other hand, we may use, similarly as in §4.1, a three-pronged discretization method. First, integration against the weight function is "regularized" by means of the change of variable $x \mapsto x^3/3$; thus,

$$\int_0^\infty p(x)w(x)dx = \frac{2^{\frac{2}{3}}\pi}{3^{\frac{1}{6}}\Gamma(\frac{2}{3})} \int_0^\infty p(\tfrac{1}{3}x^3)\mathrm{Ai}(2^{-\frac{2}{3}}x^2)\cdot e^{-x^3/3}dx. \tag{4.12}$$

The discretization is effected by using the decomposition $\mathbb{R}_+ = (0,2]\cup[2,6]\cup[6,\infty)$ and N-point Gauss-Legendre quadrature on the first two intervals, and N-point Gauss quadrature relative to the weight function $e^{-x^3/3}$, $0 < x < \infty$, on the last interval (after transforming the integral over $[6,\infty)$ to one over $[0,\infty)$). The latter quadrature rules have been generated by a "general-purpose" discretization method [4, p. 95]; see also [2]. In this way, when $n = 10$, then $N = 51$ (in double precision) yields about 12-digit accuracy, $N = 91$ (in quadruple precision) an accuracy of about 29 digits. For $n = 40$, the respective numbers are $N = 81$ and $N = 161$.

Values of the recursion coefficients α_k, β_k to 28 decimal digits are given in Appendix A2 for $k = 0, 1, \ldots, 39$. These again permit the generation of Gauss and Gauss-Radau quadrature rules with up to 40 points. Double- and quadruple-precision fortran programs producing these coefficients are accessible at `http://www.cs.purdue.edu/archives/2001/wxg/codes/` in the files `d0Pairy.f` and `q0Pairy.f`. The coefficients themselves to 28 decimals can be found in the file `coeffairy` at the same URL.

Acknowledgement. The author gratefully acknowledges helpful comments from two anonymous referees.

Appendix A1: Recursion coefficients α_k and β_k for the orthogonal polynomials relative to the weight function $\frac{\sqrt{3}}{\pi} K_{\frac{1}{3}}(x)$, $0 < x < \infty$.

TABLE 1

Recursion coefficients for orthogonal polynomials with Bessel weight function

k	dalpha(k)	dbeta(k)
0	0.577350269189625764509148780 5D+00	0.1000000000000000000000000000D+01
1	0.2540341184434353363840254634D+01	0.5555555555555555555555555556D+00
2	0.4530325099277556972940619392D+01	0.3080000000000000000000000000D+01
3	0.6525263205367024754584421146D+01	0.7597308896010194711493412792D+01
4	0.8522091681572316495410571818D+01	0.1411128895064359048476764685D+02
5	0.1051987069032579681709327397D+02	0.2262328375651997214118489043D+02
6	0.1251820534200411842698661310D+02	0.3313393449511827909015205175D+02
7	0.1451689745070847589982540347D+02	0.4564360198457644797420242203D+02
8	0.1651583533890971294501705203D+02	0.6015251169387165444574183086D+02
9	0.1851495070370021650146786956D+02	0.7666081512279027237029582199D+02
10	0.2051419913459637659996186934D+02	0.9516861964153151276420251970D+02
11	0.2251355035096639733155330055D+02	0.1156760045129536402241437743D+03
12	0.2451298290692544758973945221D+02	0.1381830301661429736830519992D+03
13	0.2651248113403138074584401742D+02	0.1626897438888321271102896829D+03
14	0.2851203328421788988619681952D+02	0.1891961834909268127004429781D+03
15	0.3051163035337998449498607362D+02	0.2177023797559084152522105542D+03
16	0.3251126530928971516451325688D+02	0.2482083581355438536773868170D+03
17	0.3451093256931540393887725062D+02	0.2807141399544099117867506745D+03
18	0.3651062763776024473491489160D+02	0.3152197432866833050342266185D+03
19	0.3851034684821817685385767070D+02	0.3517251836077464523819711369D+03
20	0.4051008717681441792603844069D+02	0.3902304742873398067597234709D+03
21	0.4250984610438625709490911393D+02	0.4307356269688528086704175367D+03
22	0.4450962151314044127608953558D+02	0.4732406518652590307625711171D+03
23	0.4650941160804019999493643228D+02	0.5177455579930047052041031595D+03
24	0.4850921485622134798647588526D+02	0.5642503533590167393634950613D+03
25	0.5050902993974754185328926380D+02	0.6127550451118075040591250302D+03
26	0.5250885571836803795271883128D+02	0.6632596396647438874918855282D+03
27	0.5450869119986848180007865562D+02	0.7157641427974924632092938047D+03
28	0.5650853551625092002779691310D+02	0.7702685597401779256832448017D+03
29	0.5850838790443559309423523307D+02	0.8267728952437190489066061745D+03
30	0.6050824769050409076746501085D+02	0.8852771536390157066778742762D+03
31	0.6250811427674077725581328007D+02	0.9457813388870707655384939225D+03
32	0.6450798713090365394190828550D+02	0.1008285454621685966632728510D+04
33	0.6650786577728518444594287897D+02	0.1072789504186032141840176389D+04
34	0.6850774978922060972058434023D+02	0.1139293490664133569567757833D+04
35	0.7050763878277471470954744945D+02	0.1207797416908104118798814682D+04
36	0.7250753241139409466091921010D+02	0.1278301285561814669842441005D+04
37	0.7450743036135516422801459683D+02	0.1350805099081546598221771377D+04
38	0.7650733234787168021992493344D+02	0.1425308859754087068026168299D+04
39	0.7850723811175176515929034805D+02	0.1501812569712642681574641394D+04

Appendix A2: Recursion coefficients α_k and β_k for the orthogonal polynomials relative to the weight function $\frac{2^{2/3}\pi}{3^{5/6}\Gamma(2/3)}\, x^{-\frac{2}{3}} e^{-x} \mathrm{Ai}((\frac{3}{2}x)^{\frac{2}{3}})$, $0 < x < \infty$.

TABLE 2

Recursion coefficients for orthogonal polynomials with Airy weight function

k	dalpha(k)	dbeta(k)
0	0.142857142857142857142857142857D+00	0.100000000000000000000000000000D+01
1	0.111050883021507256513376464 4D+01	0.675039246467817896389324960 8D-01
2	0.210392326230336646971904665 0D+01	0.617643552063145494647736632 2D+00
3	0.310061292361946806020404174 9D+01	0.166522236957972876399359810 4D+01
4	0.409852299005335030025967735 6D+01	0.321166543793381330144267310 8D+01
5	0.509704801791223795221119136 6D+01	0.525743979142826699856989400 3D+01
6	0.609593477214273528940506765 1D+01	0.780276287181614124507891623 0D+01
7	0.709505574261517496969576111 5D+01	0.108477558276633146373178521 3D+02
8	0.809433870497979030457252285 3D+01	0.143924940512795030089198295 7D+02
9	0.909373924746299313669996612 3D+01	0.184370281292212892877309851 2D+02
10	0.100932283460995652909567492 5D+02	0.229813939105091131332010300 4D+02
11	0.110927861189942779638951302 1D+02	0.280256178765643953018951313 9D+02
12	0.120923984257352418314966796 5D+02	0.335697202370159831298507901 5D+02
13	0.130920548961377960543055266 4D+02	0.396137168256624709247676305 2D+02
14	0.140917477281966677767988646 8D+02	0.461576203191747185465891788 0D+02
15	0.150914709237836670579585726 3D+02	0.532014410521356847800320046 9D+02
16	0.160912197852817198395378806 8D+02	0.607451875804526295737673856 3D+02
17	0.170909905739694401379355849 2D+02	0.687888670819003203643093511 6D+02
18	0.180907802720898387744535183 4D+02	0.773324856478136069424985934 1D+02
19	0.190905864133403607204054403 1D+02	0.863760484999947472661023554 5D+02
20	0.200904069596767811962535999 8D+02	0.959195601549889848533759188 8D+02
21	0.210902402101782518861473383 7D+02	0.105963024550527781447927378 9D+03
22	0.220900847325546803008011137 9D+02	0.116506445144263346718727706 1D+03
23	0.230899393109333024485879620 6D+02	0.127549824991866940877246145 3D+03
24	0.240898029055397428000857145 4D+02	0.139093166809525760664641485 2D+03
25	0.250896746211977002281557546 3D+02	0.151136473024483870414514709 3D+03
26	0.260895536824542958652063512 8D+02	0.163679745816307447054029719 2D+03
27	0.270894394137443182315823629 2D+02	0.176722987150872614573511134 6D+03
28	0.280893312234295952138276178 9D+02	0.190266198808584758065276477 5D+03
29	0.290892285908492813185220733 2D+02	0.204309382407982040603628164 8D+03
30	0.300891310557319012975563905 6D+02	0.218852539425613252126254188 5D+03
31	0.310890382094763306745738599 5D+02	0.233895671212884188668065299 8D+03
32	0.320889496879238810560512015 4D+02	0.249438779010418946709190589 9D+03
33	0.330888651653291509523787070 7D+02	0.265481863960369857181168025 1D+03
34	0.340887843493015102522392702 0D+02	0.282024927117023095657919423 8D+03
35	0.350887069765377637853926155 6D+02	0.299067969455979707687035815 7D+03
36	0.360886328092037692729437040 7D+02	0.316610991882139109061573931 9D+03
37	0.370885616318514956117472495 8D+02	0.334653995236670563399239162 4D+03
38	0.380884932487803208140797683 8D+02	0.353196980303125116002560474 9D+03
39	0.390884274817688376603066089 7D+02	0.372239947812814040776766994 2D+03

REFERENCES

1. Abramowitz, M. and Stegun, I.A. (Eds.) 1964. Handbook of mathematical functions with formulas, graphs, and mathematical tables. NBS Applied Mathematics Series, Vol. 55, U.S. Government Printing Office, Washington, D.C.
2. Gautschi, W. 1983. How and how not to check Gaussian quadrature formulae. *BIT 23*, pp. 209–216.
3. Gautschi, W. 1994. Algorithm 726: ORTHPOL — a package of routines for generating orthogonal polynomials and Gauss-type quadrature rules. *ACM Trans. Math. Software 20*, pp. 21–62.
4. Gautschi, W. 1996. Orthogonal polynomials: applications and computation. In *Acta Numerica 1996* (A. Iserles, Ed.), Cambridge University Press, Cambridge, pp. 45–119.
5. Gordon, R.G. 1969. New method for constructing wavefunctions for bound states and scattering. *J. Chem. Phys. 51*, pp. 14–25.
6. Gordon, R.G. 1970. Constructing wavefunctions for nonlocal potentials. *J. Chem. Phys. 52*, pp. 6211–6217.
7. Gradshteyn, I.S. and Ryzhik, I.M. 2000. Tables of integrals, series, and products. Academic Press, San Diego, CA.
8. Lozier, D.W. and Olver, F.W.J. 1994. Numerical evaluation of special functions. In *Mathematics of Computation 1943–1993: a half-century of computational mathematics (Vancouver, BC, 1993)* (W. Gautschi, Ed.), Proc. Sympos. Appl. Math., v. 48, Amer. Math. Soc., Providence, RI, 1994, pp. 79–125.
9. Schulten, Z., Anderson, D.G.M., and Gordon, R.G. 1979. An algorithm for the evaluation of the complex Airy functions. *J. Computational Phys. 31*, pp. 60–75.
10. Temme, N.M. 1975. On the numerical evaluation of the modified Bessel function of the third kind. *J. Computational Phys. 19*, pp. 324–337.
11. Watson, G.N. 1958. A treatise on the theory of Bessel functions, 2nd ed., Cambridge University Press, Cambridge.
12. Wong, R. 1982. Quadrature formulas for oscillatory integral transforms. *Numer. Math. 39*, pp. 351–360.